In fact, the mathematical model, due to its complexity, was solved by a UNIVAC 1180 computer. Some results are presented in Fig. 2.

CONCLUSIONS

In our experimental apparatus, the overall sensitivity is $0{\cdot}75\ \mathrm{cm}^3/^\circ\mathrm{K}$ at boiling water temperature of 100°C (the displacement corresponding to $1\ \mathrm{cm}^3$ is 1·8 cm), while the intrinsic inaccuracy, shown by small oscillations of the liquid level of the azotometer, is in the order of $0{\cdot}05\ \mathrm{cm}^3$. Thus the procedure seems perfectly adequate for an accurate measurement of the heat transfer coefficient in pool boiling, given by:

$$h_B = \frac{W - W_{\mathrm{diss}}}{2\pi R_3 L(T_{\mathrm{PE}} - T_{\mathrm{BW}})} \tag{9}$$

even if $(T_{\mathrm{PE}} - T_{\mathrm{BW}})$ is a few degrees. On the contrary the intrinsic uncertainty is caused by the irreproducibility of the boiling phenomenon itself.

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LIMITING NUSSELT NUMBERS IN FINITE MHD DUCTS

WILLIAM J. THOMSON and TIMOTHY H. TSAI

Dept. of Chemical Engineering, University of Idaho, Moscow, Idaho, U.S.A.

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NOMENCLATURE

B, magnetic flux density;
C_p, heat capacity;
D_e, equivalent diameter $[4a/(1 + 1/\gamma)]$;
E, electric field;
Ec, Eckert number $[\bar{u}^2/C_p(T_c - T_w)]$;
H^*, dimensionless magnetic field,

$$\left[H_x/a^2\left(-\frac{\partial P}{\partial x}\right)\left(\frac{\sigma_c}{M_c}\right)^{\frac{1}{2}}\right];$$

J, current density;
k, thermal conductivity;
M, Hartmann number $[aB_0(\sigma_c/M_c)^{\frac{1}{2}}]$;
N_E, electric field parameter $[E_0/\bar{u}B_0]$;
Nu, Nusselt number $[q_w D_e/k(\bar{T} - T_w)]$;
P, pressure;
q, heat flux;
u^*, dimensionless velocity, $[\mu_c u/a^2(-\partial P/\partial x)]$;
y^*, Y/z;
Z^*, Z/b.

Greek symbols
γ, aspect ratio of duct (b/a);
θ, dimensionless temperature, $T - T_w/T_c - T_w$;
μ, viscosity;
ρ, density;
σ, electrical conductivity.

Subscripts
c, centerpoint of duct;
w, wall.

Superscripts
$'$, "pseudo" parameter defined on basis of a velocity other than the average velocity;
$\bar{\ }$, average value;
$*$, dimensionless variable.

1. INTRODUCTION

THERE have been many analyses of MHD heat transfer in parallel plate geometries [1–5] but studies in finite ducts have been relatively scarce. Despite the abundant work on the simpler parallel plate problems, there has been surprisingly little emphasis placed on predictions of the Nusselt number. Finite ducts have not received as much attention due in part to the complexity of the problem since the fluid flow is influenced by the nature of recirculating currents which must be accounted for by relating the local current density to the magnetic field which is induced in the direction of the flow. This is particularly important when the duct walls are electrical insulators and it then becomes necessary to solve two coupled partial differential equations: the momentum equation and the equation describing the distribution of the induced magnetic field.

In the more general case where heat transfer is also important, three coupled partial differential equations must be solved unless the energy equation can be decoupled by virtue of constant fluid properties. The problem of fully developed heat transfer in this situation has been considered by Eraslan and Snyder [6] and also by Singer [7]. Both analyses were conducted under the assumption of constant fluid properties with Eraslan and Snyder presenting their results in the form of isotherms for boundary conditions of constant heat flux at the walls. Singer obtained his solution in the form of a system of infinite series but of such complexity that he did not attempt to calculate numerical results.

To date there has been no reported attempt to approach the electrically insulated duct problem in such a manner as to yield direct calculations of limiting Nusselt numbers. In addition there has been no consideration of the effects of either variable fluid properties or of *externally* applied electric fields on the limiting Nusselt numbers in finite MHD ducts. Thus, the problem which is considered here is oriented towards predictions of limiting Nusselt numbers under conditions of variable fluid properties for MHD flow in finite rectangular ducts. Consideration is also given to the effects of external electric fields under conditions where the electric field can be assumed to be constant within the fluid.

2. ANALYSES

2.1 *General case*

For the general case of a rectangular MHD duct but under the assumption of fully developed conditions in the x-direction, the momentum equation reduces to

$$0 = -\frac{\partial P}{\partial x} + \frac{\partial}{\partial y}\left(\mu\frac{\mathrm{d}u_x}{\mathrm{d}y}\right) + \frac{\partial}{\partial z}\left(\mu\frac{\mathrm{d}u_x}{\mathrm{d}z}\right) - (\boldsymbol{J}\times\boldsymbol{B})_x = 0. \quad (1)$$

Where it has been assumed that the fluid is Newtonian and the flow is laminar and steady. The current density vector, $\boldsymbol{J}$, can be given either by Ohm's Law

$$\boldsymbol{J} = \sigma[\boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B}] \quad (2)$$

or by Ampere's Law

$$\boldsymbol{J} = \nabla\times\boldsymbol{H}. \quad (3)$$

In general the y and z-components of the electric field vector are not constants and in such cases it is more convenient to express $\boldsymbol{J}$ in terms of equation (3). The momentum equation then becomes

$$0 = -\frac{\partial P}{\partial x} + \frac{\partial}{\partial y}\left(\mu\frac{\partial u_x}{\partial y}\right) + \frac{\partial}{\partial z}\left(\mu\frac{\partial u_x}{\partial z}\right) + B_0\frac{\partial H_x}{\partial y}. \quad (4)$$

To complete the mathematical description of the flow it is necessary to generate an expression which describes the variation of the induced magnetic field, H_x, in the y–z plane. Combining Maxwell's equations and Ohm's Law this equation simplifies to

$$\frac{\partial}{\partial y}\left(\frac{1}{\sigma}\frac{\partial H_x}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{1}{\sigma}\frac{\partial H_x}{\partial z}\right) + B_0\frac{\partial u_x}{\partial y} = 0. \quad (5)$$

In equations (4) and (5) allowance has been made for possible variations in the viscosity and the electrical conductivity of the fluid. For many problems of practical interest the fluid properties are functions of temperature thereby necessitating the coupling of the energy equation with both equations (4) and (5).

To derive the energy equation we restrict ourselves to the limiting case of fully developed thermal flow under conditions of a constant wall heat flux per unit axial length of duct. Neglecting viscous dissipation and heat conduction in the axial direction the energy equation can be written for a constant thermal conductivity as

$$\rho C_p u_x\frac{\partial T}{\partial x} = k\left[\frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right] + \frac{1}{\sigma}\left[\left(\frac{\partial H_x}{\partial y}\right)^2 + \left(\frac{\partial H_x}{\partial z}\right)^2\right]. \quad (6)$$

It can be shown that, for fully developed heat transfer with a constant heat flux, $\partial T/\partial x$ is a constant in equation (6) and can be related to the Nusselt number by taking an energy balance over an infinitesimal volume of the duct. It should also be noted that under these conditions the temperature, and hence the fluid properties, will vary in the axial direction. Thus, implicit in the derivation of equations (4)–(6) is the assumption that axial temperature gradients are small with respect to those in the y–z plane.

The applicable boundary conditions at the wall are

$$u_x = 0 \quad \frac{\partial H_x}{\partial n} = 0 \text{ (Perfect electrical insulator)}$$

$$T = T_w$$

$$\frac{\partial H_x}{\partial t} = 0 \text{ (Perfect electrical conductor)}.$$

2.2 *The electrically insulated duct*

It is worthwhile to consider the situation where all walls are perfect electrical insulators since in this case, $(H_x)_w = 0$.

Furthermore, in this situation the only currents are *induced* currents and unless magnetic interactions are large, joule heating becomes negligible.

In dimensionless form, the resulting equations are as follows

$$\frac{\partial}{\partial y^*}\left(\mu^*\frac{\partial u_x^*}{\partial y^*}\right) + \frac{1}{\gamma^2}\frac{\partial}{\partial z^*}\left(\mu^*\frac{\partial u_x^*}{\partial z^*}\right) + M_c\frac{\partial H_x^*}{\partial y^*} + 1 = 0 \quad (7)$$

$$\frac{\partial}{\partial y^*}\left(\frac{1}{\sigma^*}\frac{\partial H_x^*}{\partial y^*}\right) + \frac{1}{\gamma^2}\frac{\partial}{\partial z^*}\left(\frac{1}{\sigma^*}\frac{\partial H_x^*}{\partial x^*}\right) + M_c\frac{\partial u_x^*}{\partial y^*} = 0 \quad (8)$$

$$\left[\frac{Nu(1+1/\gamma)^2}{4\bar{\theta}\bar{u}^*}\right]u_x^* = \frac{\partial^2\theta}{\partial y^{*2}} + \frac{1}{\gamma^2}\frac{\partial^2\theta}{\partial z^{*2}}. \quad (9)$$

Equations (7)–(9) must be solved simultaneously, subject to the specification of the temperature dependency of μ and σ and choosing the coefficient of u_x^* in equation (9) such that $\theta_c = 1$. For simplicity, it is assumed that the fluid properties vary according to power law functions; that is,

$$\mu^* = \theta^\eta, \qquad \sigma^* = \theta^\omega. \tag{10}$$

The values of η and ω depend upon the nature of the fluid in question as well as upon the temperature range which exists in the duct.

2.3 *Externally applied electric fields*

If the walls at $z^* = \pm 1$ are electrical conductors, then energy can be either extracted or inputted to the duct depending on the electrical circuit external to the duct. For our purposes we assume that recirculating currents are negligible with respect to those passing through the conducting walls and thus, the only component of the electric field vector is E_z, which is constant. In this case joule heating cannot be ignored and the pertinent differential equations are

$$\frac{\partial}{\partial y^*}\left(\mu^* \frac{\partial u_x^*}{\partial y^*}\right) + \frac{1}{\gamma^2}\frac{\partial}{\partial z^*}\left(\mu^* \frac{\partial u_x^*}{\partial z^*}\right) + M_c^2[N_E' - u_x^*]\,\sigma^* + 1 = 0 \tag{11}$$

$$\left[\frac{N_u(1 + 1/\gamma)^2}{4\bar{\theta}\overline{u_x^*}}\right] u_x^* = \frac{\partial^2\theta}{\partial y^{*2}} + \frac{1}{\gamma^2}\frac{\partial^2\theta}{\partial z^{*2}} + M_c^2 PrEc'(u_x^* - N_E')^2\,\sigma^*. \tag{12}$$

3. RESULTS

The pertinent equations for both cases were solved by finite difference numerical methods using a Modified Gauss–Seidel algorithm. Values of η and ω were chosen according to estimates of how the properties of actual MHD fluids would vary with temperature and only Hartmann numbers less than 10 were considered. Limiting Nusselt numbers were calculated for situations representing an MHD accelerator ($N_E \backsim 1{\cdot}5$) and for an MHD generator ($N_E \backsim 0{\cdot}6$) with the product of the Eckert and Prandtl numbers held constant at 0·1.

As shown in Fig. 1 the general trend is for the Nusselt number to increase with aspect ratio since as the aspect ratio increases the temperature profile becomes flatter and D_e increases. The influence of joule heating is manifested not only by the large differences in the results for the accelerator as contrasted to the generator but also by the fact that the Nusselt numbers are lower for the accelerator than for the insulated duct. This latter result is due to the tendency of joule heating to lower the Nusselt number, a result which has been reported by previous authors [4, 5]. However, the occurrence of much lower Nusselt numbers in the generator case requires a more detailed analysis.

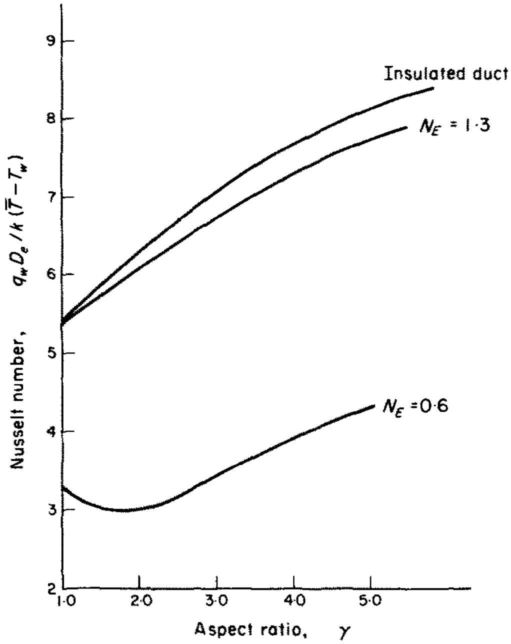

FIG. 1. Variations of Nusselt numbers with aspect ratio; constant properties ($M_c = 5$).

The explanation lies in the competing influences of the convective and joule heating contributions to the energy balance. For MHD generators the $\boldsymbol{J} \times \boldsymbol{B}$ force is a retarding force and for a given pressure drop, $\overline{u_x^*}$ will decrease as the magnitude of the current which is drawn through the external circuit increases. Since $\overline{u_x^*}$ appears in *both* the convective and joule heating terms in equation (12), the net effect is that the combination of reduced convection and increased joule heating produce lower Nusselt numbers. However, the $\boldsymbol{J} \times \boldsymbol{B}$ force is an accelerating force in the MHD accelerator and thus the increased convection more than compensates for the increased joule heating and higher Nusselt numbers are the result.

Consistent with the above discussion it was also found that the Nusselt number was not strongly dependent on the Hartmann number for either the accelerator or the insulated duct. However, since increasing Hartmann numbers produce larger values of both the $\boldsymbol{J} \times \boldsymbol{B}$ force and the joule heating, the Nusselt number decreases sharply with increasing Hartmann numbers for the MHD generator. A separate evaluation of joule heating effects was also accomplished by studying the effects of variations in $PrEc$. As expected, increases in joule heating resulted in decreased Nusselt numbers for all cases considered with the effect being more pronounced for generators.

Figure 2 shows the results obtained when the electrical conductivity of the fluid was allowed to vary according to the power law assumption of equation (10). These results are for cooled walls and for a fluid with an electrical conductivity which decreases with decreasing temperature. Since the Nusselt number increases with increasing Hartmann number in constant property flow for both MHD accelerators and insulated ducts, it would be expected that the

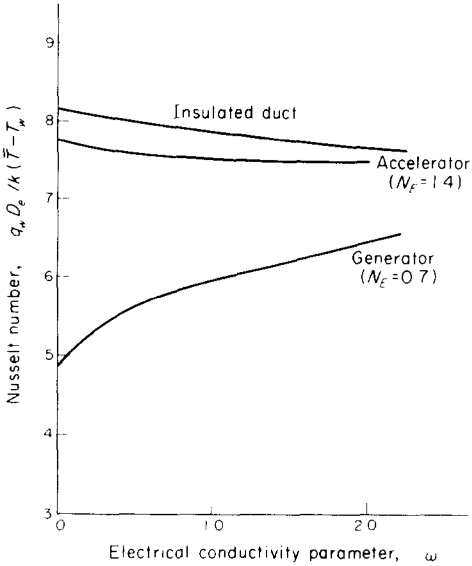

FIG. 2. The effect of variable electrical conductivity on Nusselt number ($M_c = 5$, $\gamma = 5$, constant visocosity).

Nusselt number would decrease as the electrical conductivity variance increases in these same flows. This is due to the fact that for cooled walls the local value of the Hartmann number decreases as the walls are approached. Thus the overall effect is to have an effective Hartmann number which is less than that calculated on the basis of the centerpoint conditions. It should be noted however, that as Thomson [9] has shown, it is not possible to account for local Hartmann number variations in terms of averaged bulk properties. The generator results shown in Fig. 2 have the opposite dependency on the conductivity variation because, as previously shown, it has the opposite dependency on Hartmann number for constant property flows.

It is also noteworthy that small changes in the conductivity parameter produce significant changes in the Nusselt number results. Thus even though the conductivity variation may be slight in a given situation, the Nusselt number can still be in significant error if evaluated on the basis of constant fluid properties. This is particularly true for the generator due to the additive influences of convection and joule heating.

Results were also obtained for situations where the viscosity of the fluid was also temperature dependent. For all cases considered, accounting for viscosity variations influenced the Nusselt number results in much the same way as they would in ordinary hydrodynamic flow. The maximum discrepancy between Nusselt numbers calculated on the basis of constant fluid properties and those calcu- calculated by taking viscosity variations into account was found to be 15 per cent.

4. CONCLUSIONS

It has been shown that the magnitude and character of the limiting Nusselt numbers in MHD generators are very different than those for either MHD accelerators or electrically insulated ducts. The generator results can be explained in terms of the competing influences of convection and joule heating on the differential energy balance since these contributions are additive for generators whereas they are compensatory in accelerators. The Nusselt number calculations for variable property MHD flows show significant departures from the constant property results. The generators are more sensitive to variations in the electrical conductivity, exhibiting up to 20 per cent changes in the Nusselt number even for small property variations. It was further concluded that viscosity variations were as important as they would be in ordinary hydrodynamic flow but that there was no *separate* dependence on the electromagnetic environment.

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